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COMMENT

On the resonance effect of the nonlinear string equation

A Kh Fridman

Heat and Mass Transfer Institute, BSSR Academy of Sciences, Minsk, USSR

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Abstract. Based on the third-order operator factorisation, the resonance interaction of solitons for the nonlinear string equation with a positive dispersion term is considered and operators for creation and annihilation of a resonance triad are constructed.

Let us consider the nonlinear string equation (Zakharov and Shabat 1974)

$$\frac{3}{4}\beta^2\frac{\partial^2 u}{\partial t^2}+\lambda\frac{\partial^2 u}{\partial x^2}+\frac{1}{4}\frac{\partial^4 u}{\partial x^4}+\frac{3}{4}\frac{\partial^2}{\partial x^2}(u^2)=0 \tag{1}$$

which allows the Lax representation

$$\beta\ \partial L/\partial t=LA-AL \tag{2}$$

where

$$L=\frac{\partial^3}{\partial x^3}+\frac{3}{2}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}+\frac{3}{4}\frac{\partial^2\phi}{\partial x^2}-\frac{3}{4}\beta\frac{\partial\phi}{\partial t}+\lambda\frac{\partial}{\partial x}, \tag{3}$$

$$A=\partial^2/\partial x^2+\partial\phi/\partial x, \qquad u=\partial\phi/\partial x. \tag{4}$$

The second-order operator factorisation (Fridman and El'yashkevich 1979) has been considered that permits determination of the operators of creation and annihilation of solitons and the scattering operator for collision of an arbitrary wave disturbance with a soliton.

The present paper realises this method for the third-order operator (3).

Consider factorisation of the L operator

$$L_0-\alpha=H_{01}H_{10} \tag{5}$$

where

$$H_{10}=\partial/\partial x+\partial v/\partial x, \tag{6}$$

$$H_{01}=\frac{\partial^2}{\partial x^2}-\frac{\partial v}{\partial x}\frac{\partial}{\partial x}-\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2-\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+\frac{3}{2}\beta\frac{\partial v}{\partial t}+\lambda. \tag{7}$$

Expansion of equations (5)–(7) allows us to build the following equation coupled with (1),

$$3\beta^2\frac{\partial^2 v}{\partial t^2}+4\lambda\frac{\partial^2 v}{\partial x^2}+\frac{\partial^4 v}{\partial x^4}-6\left(\frac{\partial v}{\partial x}\right)\frac{\partial^2 v}{\partial x^2}+6\beta\frac{\partial v}{\partial t}\frac{\partial^2 v}{\partial x^2}=0, \tag{8}$$

for which the Lax representation (2) may be realised by the $(\tilde{L}, \tilde{A})$ pair

$$\tilde{L} = \begin{pmatrix} 0 & H_{10} \\ H_{01} & 0 \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \tag{9}$$

where A_0 and A_1 have the form of the A operator for the initial equation and depend on

$$\frac{\partial \phi_0}{\partial x} = \frac{\partial^2 v}{\partial x^2} - \left(\frac{\partial v}{\partial x}\right)^2 + \beta \frac{\partial v}{\partial t}, \tag{10}$$

$$\frac{\partial \phi_0}{\partial t} = \frac{\partial^2 v}{\partial x\, \partial t} - 2\frac{\partial v}{\partial t}\frac{\partial v}{\partial x} - (3\beta)^{-1}\frac{\partial^2 v}{\partial x^3} + 2(3\beta)^{-1}\left(\frac{\partial v}{\partial x}\right)^3 - 4\lambda(3\beta)^{-1}\frac{\partial v}{\partial x} - \alpha, \tag{11}$$

$$\frac{\partial \phi_1}{\partial x} = -\frac{\partial^2 v}{\partial x^2} - \left(\frac{\partial v}{\partial x}\right)^2 + \beta \frac{\partial v}{\partial t}, \tag{12}$$

$$\frac{\partial \phi_1}{\partial t} = -\frac{\partial^2 v}{\partial x\, \partial t} - 2\frac{\partial v}{\partial t}\frac{\partial v}{\partial x} - (3\beta)^{-1}\frac{\partial^3 v}{\partial x^3} + 2(3\beta)^{-1}\left(\frac{\partial v}{\partial x}\right)^3 - 4\lambda(3\beta)^{-1}\frac{\partial v}{\partial x} - \alpha. \tag{13}$$

Expressions (10)–(13) follow directly from the representation

$$\partial \tilde{L}^2/\partial t = \tilde{L}^2\tilde{A} - \tilde{A}\tilde{L}^2 \tag{14}$$

and are recognised as the Miura transformations which couple two solutions of equation (1) with the solution of equation (8). This transformation can be written in the operator form

$$L_0 - \alpha = H_{01}H_{10}, \qquad L_1 - \alpha = H_{10}H_{01}. \tag{15}$$

Elimination of v in equations (10)–(13) gives the Bäcklund transformation for equation (1).

As in the case of the second-order operator the former solution $\partial\phi_0/\partial x$, $\partial\phi_0/\partial t$ does not determine unambiguously the solution of equation (8); however, the asymptote $\beta\sqrt{3}\,\partial v/\partial x$ (the multiplier $\beta\sqrt{3}$ is taken for convenience) is determined now by the third-order algebraic equation

$$k^3 - \tfrac{1}{4}k - \alpha = 0. \tag{16}$$

Later on two cases are considered,

$$(1)\ \beta^2 = 3, \lambda = -\tfrac{1}{4}; \qquad (2)\ \beta^2 = -3, \lambda = \tfrac{1}{4}, \tag{17}$$

the difference between which is attributed to the sign of the dispersion term. In the case of a positive dispersion term for $|\alpha| < 1/6\sqrt{12}$ we number the roots of equation (16) as follows: $k_1 < k_2 < k_3$. Now let us choose any pair of the roots of equation (16) k_i, k_j. Without loss of generality we assume that the former solution does not involve solitons. A further consideration can be carried out similarly to the Korteweg–de Vries equation. Consider three different solutions of equation (8) having different asymptotes at infinity,

$$\begin{aligned} &(1)\ \frac{\partial v^{\text{in}}}{\partial x}(\pm\infty, t) = k_i, \qquad &&(2)\ \frac{\partial v^{\text{out}}}{\partial x}(\pm\infty, t) = k_j, \\ &(3)\ \frac{\partial v}{\partial x}(-\infty, t) = k_i, &&\frac{\partial v}{\partial x}(+\infty, t) = k_j. \end{aligned} \tag{18}$$

Such ambiguity of the Miura transformation yields three possible factorisations of the L operator

$$L_0-\alpha = H_{01}^{\text{in}}H_{10}^{\text{in}} = H_{01}^{\text{out}}H_{10}^{\text{out}} = H_{01}H_{10}, \tag{19}$$

$$L_1^{\text{in}}-\alpha = H_{10}^{\text{in}}H_{01}^{\text{in}}, \tag{20}$$

$$L_1^{\text{out}}-\alpha = H_{10}^{\text{out}}H_{01}^{\text{out}}, \tag{21}$$

$$L_1-\alpha = H_{10}H_{01}. \tag{22}$$

$\partial\phi_1/\partial x$ has the following asymptotic form at $t\to\mp\infty$·

$$\frac{\partial\phi_1}{\partial x}\to\begin{cases}\dfrac{\partial\phi^{\text{in}}}{\partial x}+\dfrac{(k_i-k_j)^2}{4}\operatorname{sech}^2\dfrac{(k_i-k_j)}{2}(x-\sqrt{3}k_l t), & t\to-\infty, \quad (23)\\ & i\neq j\neq l\\ \dfrac{(k_i-k_j)^2}{4}\operatorname{sech}^2\dfrac{(k_i-k_j)}{2}(x-\sqrt{3}k_l t-\delta)+\dfrac{\partial\phi^{\text{out}}}{\partial x}, & t\to+\infty, \quad (24)\end{cases}$$

and describes the process of collision of the disturbance $\partial\phi^{\text{in}}/\partial x$ with a soliton of equation (1). The coefficients $\partial\phi_1^{\text{in}}/\partial x$ and $\partial\phi_1^{\text{out}}/\partial x$ of the operators L_i^{in} and L_1^{out} have the asymptotic forms

$$\partial\phi_1^{\text{in}}/\partial x\to\partial\phi^{\text{in}}/\partial x, \qquad t\to-\infty, \tag{25}$$

$$\partial\phi_1^{\text{out}}/\partial x\to\partial\phi^{\text{out}}/\partial x, \qquad t\to+\infty. \tag{26}$$

The unitary scattering operator S which describes the process of collision of the disturbance $\partial\phi^{\text{in}}/\partial x$ with a soliton is determined similarly to the Korteweg–de Vries equation,

$$S\psi_1^{\text{in}} = \psi_1^{\text{out}} \tag{27}$$

where

$$L_1^{\text{in}}\psi_1^{\text{in}} = \alpha\psi_1^{\text{in}}, \tag{28}$$

$$L_1^{\text{out}}\psi_1^{\text{out}} = \alpha\psi_1^{\text{out}}. \tag{29}$$

From (19)–(29) the equivalent definition follows:

$$SH_{\text{in}} = H_{\text{out}}. \tag{30}$$

In the case of a negative dispersion term at $|\alpha|>1/6\sqrt{12}$ a pair of conjugate roots of equation (16) is chosen. In other respects the process of collision is described similarly.

The factorisation (5) allows the following generalisation,

$$(L_0-\alpha_1)(L_0-\alpha_2)\ldots(L_0-\alpha_n) = H_{0n}H_{n0}, \tag{31}$$

$$(L_n-\alpha_1)(L_n-\alpha_2)\ldots(L_n-\alpha_n) = H_{n0}H_{0n}, \tag{32}$$

$$H_{n0} = H_{n\,n-1}H_{n-1\,n-2}\ldots H_{10}, \tag{33}$$

$$H_{0n} = H_{01}\ldots H_{n-2\,n-1}H_{n-1\,n}, \tag{34}$$

$$H_{i\,i-1} = \partial/\partial x + \partial v_{i-1}/\partial x, \tag{35}$$

where $\partial v_{i-1}/\partial x$ in the case of a positive dispersion term has the asymptote (k_1^i, k_2^i) or (k_2^i, k_3^i) composed of the roots of equation (16) at $\alpha=\alpha_i$ with $\max(k_1^i, k_2^i)\leqslant \min(k_2^j, k_3^i)$ for all (i, j). In the case of a negative dispersion term the asymptote is prescribed by the pair of conjugate roots. If the L_0 operator has no solutions then the index 'i' describes the number of solitons in the system

$$H_{i\,i-1}\psi_{i-1} = \psi_i, \qquad H_{i-1\,i}\psi_i = \psi_{i-1}. \tag{36}$$

Then the introduced operators $H_{i\,i-1}, H_{i-1\,i}$ can be considered as those of creation and annihilation of solitons, respectively. In the case $\alpha_1 = \alpha_2 = \alpha_3 \ldots = \alpha_n = 1/6\sqrt{12}$ the solitons degenerate into rational solitons, the explicit form of which can be obtained by the limiting transition in formulae (37)–(38).

Application of formulae (10)–(13) and (31)–(35) for a vacuum $\partial\phi_0/\partial x = 0$ allows the n-soliton solution to be built, which after simple algebraic manipulations can be reduced to the following form (Fridman 1978).

$$\frac{\partial\phi}{\partial x} = -2\frac{\partial^2}{\partial x^2}\ln f(x, t), \tag{37}$$

$$f = \sum_{\mu=0,1}\exp\left(\sum_{i<j}^{n} A_{ij}\mu_i\mu_j + \sum_{i=1}^{n}\mu_i\eta_i\right), \tag{38}$$

$$A_{ij} = \frac{(k_l^i - k_l^j)(k_m^i - k_m^j)}{(k_l^i - k_m^j)(k_m^i - k_l^j)}, \qquad l \neq m \neq n,$$

$$\eta_i = k_i(x - v_i t) + \eta_i^0, \qquad k_i = k_l^i - k_m^i, \qquad v_i = \sqrt{3}k_n^i.$$

In contrast to the Korteweg–de Vries equation, equation (1) with a positive dispersion term permits resonance interaction of solitons (Fridman 1978) when two solitons submerge into one or *vice versa*. In this case $A_{n\,n-1}$ vanishes and

$$\alpha_n = \alpha_{n-1}, \qquad k_2^{n-1} = \max(k_1^i, k_2^i) = k_2^n = \min(k_2^j, k_3^j).$$

Designate the corresponding three solutons of equation (16) at $\alpha = \alpha_n$ in terms of k_1^n, k_2^n, k_3^n. Then at $t \to -\infty$ there are two solitons, with one moving at the velocity $\sqrt{3}k_1^n$ and the other at $\sqrt{3}k_3^n$. At $t \to +\infty$ these two solitons submerge into one moving at the velocity $\sqrt{3}k_2^n$. The process of vanishing $A_{n\,n-1}$ is written as a resonance condition

$$\begin{aligned}&(k_2^n - k_1^n) + (k_3^n - k_2^n) = (k_3^n - k_1^n),\\ &\sqrt{3}k_3^n(k_2^n - k_1^n) + \sqrt{3}k_1^n(k_3^n - k_2^n) = \sqrt{3}k_2^n(k_3^n - k_1^n).\end{aligned} \tag{39}$$

In the case of resonance interaction the $H_{n-1\,n}$ operator allows the following factorisation,

$$H_{n-1\,n} = \left(\frac{\partial}{\partial x} - \frac{\partial v_n}{\partial x} - \frac{\partial v_{n-1}}{\partial x}\right)\left(\frac{\partial}{\partial x} + \frac{\partial v_n}{\partial x}\right), \tag{40}$$

where the product of operators

$$H^{\mathrm{R}}_{n\,n-2} = \left(\frac{\partial}{\partial x} + \frac{\partial v_n}{\partial x}\right)\left(\frac{\partial}{\partial x} + \frac{\partial v_{n-1}}{\partial x}\right) \tag{41}$$

is the operator of creation of a resonance triad, while

$$H^{\mathrm{R}}_{n-2\,n} = \left(\frac{\partial}{\partial x} - \frac{\partial v_n}{\partial x} - \frac{\partial v_{n-1}}{\partial x}\right) \tag{42}$$

is the operator of annihilation of the resonance triad. Not only successive generation of solitons (k_1^n, k_2^n) and (k_2^n, k_3^n) allows creation of the resonance triad. A choice of the asymptote $\partial v_n/\partial x$ in the form (k_1, k_3) does not determine $\partial v_n/\partial x$ unambiguously; one should prescribe an intermediate asymptote on the characteristic $(x-\sqrt{3}k_1t)=$ constant at $t\to-\infty$. With the asymptote being equal to k_2, the operator of creation of a resonance triad has the form of (6), and that of annihilation, the form of (7).

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